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Strong Evidence In Favor Of The Existence Of S-Matrix For Strings In Plane Waves

Dongsu Bak[1, 2] **and M. M. Sheikh-Jabbari**[1]

[1] *Department of Physics, Stanford University*
382 via Pueblo Mall, Stanford CA 94305-4060, USA

[2] *Physics Department, University of Seoul, Seoul 130-743, Korea*

`dsbak, jabbari@itp.stanford.edu`

Abstract

Field theories on the plane wave background are considered. We discuss that for such field theories one can only form $1+1$ dimensional freely propagating wave packets. We analyze tree level four point functions of scalar field theory as well as scalars coupled to gauge fields in detail and show that these four point functions are well-behaved so that they can be interpreted as S-matrix elements for 2 particle $\to$ 2 particle scattering amplitudes. Therefore, at least classically, field theories on the plane wave background do have S-matrix formulation.

1 Introduction

As a usual lore, our understanding of (perturbative) string theory is based on the S-matrix interpretation of the string scattering amplitudes, given as correlators of vertex operators and computed using the worldsheet conformal field theory. However, generally the question whether a field/string theory has an S-matrix, at least at classical level, returns to the question of having well-defined asymptotic states, i.e. freely propagating states in asymptotic regions of spacetime. For any field theory with a well-defined (positive definite) Hamiltonian, this is equivalent to having a potential that has some flat directions. However, having these asymptotic flat directions is not sufficient and one should further check that the potential at large separation falls off fast enough. In the field theory terminology and in terms of Feynman diagrams, this means that we should have off-shell particle propagation and well-behaved tree level scattering amplitudes. Then one can show that any Lorentz invariant local D ($D > 2$) dimensional field theory on flat background has S-matrix.

The question about the existence of S-matrix becomes more non-trivial for the case of field/string theories on nonflat spaces. A famous example is the string theory on the AdS space, where despite of having well-defined correlators, these correlators do not have an S-matrix interpretation. This is also manifest in the dual gauge theory in the fact that, it is conformally invariant. However in the AdS case, taking the large AdS radius limit one can show that the correlators indeed recover the S-matrix elements of flat space field theories.

Here we study existence of S-matrix for string/M theory on the plane wave background. The plane wave geometry is appearing as a Penrose limit [1] of $AdS_p \times S^q$ spaces and hence it is natural to expect strings on plane waves to have a description in terms of a specific subsector of the operators of $\mathcal{N} = 4$ SYM on $R \times S^3$, namely the large R-charge sector [2] (which hereafter will be called BMN sector). Since the appearance of the paper by Berenstein-Maldacena-Nastase (BMN) [2], there have been many papers trying to check the BMN conjecture. In these works mainly two directions have been pursued. Some have focused on the gauge theory calculations such as [3, 4] and some are devoted to the string theory side [5, 6], with the aim of reproducing the gauge theory results through string field theory amplitudes.

However, the formulation of string vertex operators in the plane wave background, as the conventional tool for doing string scatterings, has not been developed yet. As the first step to make sure that such vertex operators exist, one must have an affirmative answer to the question of existence of S-matrix for strings in plane wave background. One may try to argue that, since the plane wave geometry is coming as a Penrose limit of the AdS, similar to its AdS parent, we do not have S-matrix in this case. This may be further supported by the fact that string theory on plane wave is dual to the BMN (large R-charge) sector of a conformal gauge theory and hence the question of the existence of the S-matrix has the same answer as the AdS background. However, on the other hand, the Penrose limit is in fact a particular way of taking the large radius limit in AdS, where it is generally believed that one has a well-defined S-matrix. Moreover, since under the Penrose limit the whole causal structure as well as asymptotic behavior of the AdS is changed (e.g. now we have a one dimensional, light-like boundary) [7], it is not so obvious that the answer to the question of existence of S-matrix follows directly from the AdS parent.

Specifically here we consider field theories on a D dimensional plane wave geometry:

$$ds^2 = -2dx^+dx^- - \mu^2 \sum_{i=1}^{D-2} z_i^2 (dx^+)^2 + \sum_{i=1}^{D-2} dz_i dz_i \ . \tag{1.1}$$

The plane wave geometry produces confining potentials in the transverse directions. The shape of the potential is quadratic in the transverse coordinates for massless as well as massive particles and thus it is certainly impossible to propagate freely in the transverse directions. (In other words there is no translation symmetry along z_i directions.) Then, the only potentially flat direction is x^- and what we mean by the S-matrix formulation should be understood in this $1+1$ dimensional sense, with x^- as the spatial dimension. Hence the nature of interaction along this one spatial dimension will be the focus of our analysis in this work.

It is well-known that effective dynamics of strings is governed by field theories, in particular effective dynamics of strings in a plane wave background is governed by supergravity in the same background (for an explicit form of such an action for the axion-dilaton field of IIB supergravity see e.g. [8]). Therefore, here instead of strings on plane waves we consider field theories on the plane wave background and study the existence of S-matrix for those field theories. Since we are dealing with effective two dimensional theories, at first sight, it may seem that we are going to face the usual problem of two dimensional massless theories, namely Green's function does not fall off (in fact it grows) at infinity. However, we are in fact safe because of the leakage of the wave-function to the "transverse" directions. In particular we note that the nature of the leakage is so that the less the exchanged light-cone momentum, the more possibility for the wave-function to spread out in the transverse directions. This is one of the essential mechanisms for the existence of the S-matrix. In the following, through explicit field theory computations, we show that generic tree level amplitude for such field theories is well-behaved so that consequently these field theories admit an S-matrix formulation.

Organization of the paper is as follows. In Section 2 we consider scalar field theory with ϕ^3 and ϕ^4 interactions on the D dimensional plane wave background. Calculating the $2\phi \to 2\phi$ scattering amplitude we show that these amplitudes are smooth enough to have well-defined S-matrix description. Because of the form of the plane wave geometry (1.1) it turns out to be more convenient to use the light front coordinates. Due to peculiar features and potential problems of field theories in the light-cone gauge (in the $p_- \to 0$ limit) [9], higher spin fields should be dealt with separately. Hence in Section 3 we consider scalars coupled to vector (gauge) fields. Studying the $2\phi \to 2\phi$ amplitudes mediated by the gauge fields, we show that the gauge theories on plane wave background also admit S-matrix description, the result which we believe to be true for any generic higher spin theories including gravity. The last section contains our conclusions and remarks. In Appendix A, we have gathered some identities used in the computations and in Appendix B, through the most general two body non-relativistic quantum mechanical arguments, we have made clear how $p_- \to 0$ corresponds to the large x^- separation.

2 Scalar Field Theories on Plane Wave Background

We first consider scalar theories on the plane wave background, which would eventually reveal the essence of the question posed. With the standard kinetic term, we consider only the local interactions. The interactions are then limited to the forms of local products. We focus on the nature of tree level two particle interactions from the view point of 1+1 dimensions. Considering cubic interactions, one may build up the four point tree amplitudes corresponding to two body potentials. Thus looking at the behavior of the four point interaction, one can check whether free states in the large x^- separation are allowed or not. As we shall see below, these derived interactions fall off fast enough at large separation, so that the asymptotic free states indeed exist. The local quartic scalar interaction contributes to the four point amplitudes but, again, does not affect the existence of the asymptotically free states. The interactions of higher than fourth order do not contributes four point tree amplitude.

2.1 The model

Consider a scalar field theory on a D dimensional plane wave geometry (1.1) with the usual conventional kinetic terms:

$$\mathcal{L} = \partial_+\phi\partial_-\phi - \frac{1}{2}\mu^2 \sum_{i=1}^{D-2} z_i^2(\partial_-\phi)^2 - \frac{1}{2}(\partial_i\phi)^2 - \frac{1}{2}M^2\phi^2 - \frac{\lambda}{3!}\phi^3 . \tag{2.1}$$

where we have added a mass term, which is not typically there for a supergravity field in the plane wave background, as well as a cubic interaction term. As we can see explicitly from the Lagrangian, for a fixed μ, the above action is not invariant under Lorentz boost in the $x^\pm$ plane. Nonetheless it is quite natural to use the light-cone frame. In the following we take μ to be positive without loss of generality. The classical equation of motion for the free theory reads

$$-2\partial_+\partial_-\phi + \mu^2 \sum_{i=1}^{D-2} z_i^2\partial_-^2\phi + \partial_i^2\phi - M^2\phi = 0 \ . \tag{2.2}$$

The z_i dependence of solutions to the above equation is a $D-2$ dimensional harmonic oscillator wave function with frequency μp_-, which we will denote it by K_{n_i}, $i = 1, 2, ..., D-2$. Thus,

$$\phi = \sum_{\vec{n}} \frac{1}{2\pi} \int dp_+ \int_0^\infty \frac{dp_-}{\sqrt{2p_-}} \phi_{\vec{n}}(p_-, p_+) \prod_{i=1}^{D-2} K_{n_i}(\sqrt{\mu p_-} z_i) e^{-i(p_-x^- + p_+x^+)} + \text{h. c.} \tag{2.3}$$

solves the above equation if

$$\begin{aligned} p_+ &= \mu\, \mathcal{N} + \frac{M^2}{2p_-} \equiv \mathcal{E}(\vec{n}, p_-) \ , \\ \mathcal{N} &= (\sum_{i=1}^{D-2} n_i) + \frac{D-2}{2} \ , \end{aligned} \tag{2.4}$$

with $\{n_i\} = \vec{n}$. Here p_+ may be thought as the light-cone Hamiltonian of the above field theory.[1] Since we are in the light-cone frame, all the p_- momenta are positive definite. This

[1]It is well-known that a boost along x^- direction is in fact a scale transformation along the longitudinal direction, i.e.

$$x^- \to e^v x^-$$

where v is the hyperbolic boost angle and

$$p_- \to e^{-v} p_- , \qquad n_i \to n_i \ .$$

In the light-cone frame for the flat space, the light-cone Hamiltonian transforms as $p_+ \to e^v p_+$. However, in our case p_+ would have the usual behavior only if one also transforms $\mu \to e^v \mu$. Hence for a fixed μ the two dimensional boost invariance is lost, and for the same reason the creation operators $\phi_{\vec{n}}$ have not a simple transformation under boost. It is important to note that $p_- \to 0$ region, unlike the usual flat space light-cone frame, cannot be studied through a simple longitudinal boost.

has been made manifest in the Eq.(2.3). (As usual $p_- = 0$ in the light-cone is problematic and we shall not consider these complications.)

Because of the harmonic oscillator potential in the transverse directions (z_i dependent part), fields have a Gaussian fall-off and hence we do not have the asymptotic free states for those directions. However, the theory may effectively be treated as a two dimensional theory of component fields $\phi_{\vec{n}}$, having masses greater than or equal to $\frac{D-2}{2}\mu$. Therefore, the question about S-matrix is reduced to the question about the two dimensional effective field theory. To study the theory it would be helpful to rewrite Eq.(2.1) in terms of the $\phi_{\vec{n}}$ modes. Using the orthonormality of the K_{n_i} functions and integrating over z_i coordinates, the effective two dimensional Lagrangian becomes

$$\mathcal{L}_2 = \sum_{\vec{n}} \phi^*_{\vec{n}}(p_-, p_+)\, (p_+ - \mathcal{E}(\vec{n}, p_-))\, \phi_{\vec{n}}(p_-, p_+) \; + \mathcal{L}_{int} \;, \tag{2.5}$$

with

$$\begin{aligned}\mathcal{L}_{int} = -\lambda \sum_{\{\vec{n},\vec{m},\vec{l}\}} & \frac{1}{(p^1_- p^2_- p^3_-)^{1/2}} \phi_{\vec{n}}(p^1_-, p^1_+)\phi_{\vec{m}}(p^2_-, p^2_+)\phi^*_{\vec{l}}(p^3_-, p^3_+) \\ & \times F_{\{\vec{n},\vec{m};\vec{l}\}}(p^1_-, p^2_-; p^3_-)\; \delta(p^1_- + p^2_- - p^3_-)\delta(p^1_+ + p^2_+ - p^3_+) + \text{h. c.},\end{aligned} \tag{2.6}$$

where F is defined by

$$\begin{aligned}F_{\{\vec{n}_1,\vec{n}_2;\vec{n}_3\}}(p^1_-, p^2_-; p^3_-) &= \prod_{i=1}^{D-2} \int dz_i\, K_{n^1_i}\left(\sqrt{\mu p^1_-}\, z_i\right) K_{n^2_i}\left(\sqrt{\mu p^2_-}\, z_i\right) K_{n^3_i}\left(\sqrt{\mu p^3_-}\, z_i\right) \\ &= \int d\vec{z}\; K_{\vec{n}_1} K_{\vec{n}_2} K_{\vec{n}_3}\,.\end{aligned} \tag{2.7}$$

For the notational simplicity, we have introduced $K_{\vec{n}} \equiv \prod_{i=1}^{D-2} K_{n_i}(\sqrt{\mu p_-}\, z_i)$.

In this model, we have conserved p_-, p_+ and $SO(D-2)$ quantum numbers at each vertex. (Note that in general $\{\vec{n}\}$ are not conserved.) The conservation of $p_\pm$ is explicit due to the presence of the delta functions while the $SO(D-2)$ conservation is not and follows from the properties of $F_{\{\vec{n}_1,\vec{n}_2;\vec{n}_3\}}$. Note again that the momentum p_- takes only positive definite values while p_+ may be any real number.

2.2 Evaluating the scattering amplitude

Now to check whether the theory has an S-matrix or not, one should check if there are long-range forces in the system and study the effective forces between particles at large

Figure 1: For the vertex a), the coupling is given by $\lambda(p^1_- p^2_- p^3_-)^{-1/2} F_{\{\vec{n}_1,\vec{n}_2;\vec{n}_3\}}(p^1_-, p^2_-; p^3_-)$; for the vertex b) one has $\lambda(p^1_- p^2_- p^3_-)^{-1/2} F_{\{\vec{n}_2,\vec{n}_3;\vec{n}_1\}}(p^2_-, p^3_-; p^1_-)$. There are only two kinds of cubic vertices in the scalar theory.

x^- separation. In order that, let us focus on the simplest tree level scattering of on-shell particles and begin with the assumption that we have well-defined asymptotic free particle states. Explicitly, we consider the

$$\phi_{\vec{n}_1}\phi_{\vec{n}_2} \longrightarrow \phi_{\vec{n}_3}\phi_{\vec{n}_4}$$

scattering at tree level.

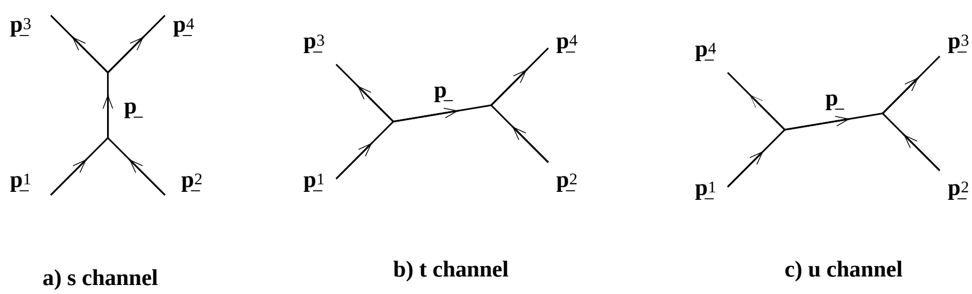

Figure 2: Four point amplitudes of the scalar theory. The t and u -channel diagrams are for $p^1_- \geq p^3_-$ and $p^1_- \geq p^4_-$, respectively.

The propagator of $\phi_{\vec{n}}$ fields reads

$$\langle\ \phi_{\vec{n}}(p_-, p_+)\phi^\dagger_{\vec{m}}(p'_-, p'_+)\ \rangle = \frac{i}{p_+ - \mathcal{E}(\vec{n}, p_-) + i\epsilon}\ \delta_{\vec{n},\vec{m}}\delta(p_+ - p'_+)\delta(p_- - p'_-)\ . \tag{2.8}$$

Due to the nonrelativistic nature of the propagator, the amplitudes are non-vanishing only when all the propagators are forwarded in x^+ direction. The two particle scattering amplitude can be in s, t or u -channels. The amplitude for s-channel is

$$\mathcal{A}^s_{2\to 2} \sim \frac{\lambda^2}{(p^1_- p^2_- p^3_- p^4_-)^{1/2}} \sum_{\vec{n}} \frac{F_{\{\vec{n}_1,\vec{n}_2;\vec{n}\}}(p^1_-, p^2_-; p_-) F_{\{\vec{n}_3,\vec{n}_4;\vec{n}\}}(p^3_-, p^4_-; p_-)}{p_- p_+ - p_- \mathcal{E}(\vec{n}, p_-)} \ , \tag{2.9}$$

where the sum is over the allowed set of n_i states,

$$\begin{aligned} p^1_- + p^2_- &= p^3_- + p^4_- = p_- \ , \\ p^1_+ + p^2_+ &= p^3_+ + p^4_+ = p_+ \ . \end{aligned} \tag{2.10}$$

As for t-channel,

$$\mathcal{A}^t_{2\to 2} \sim \frac{\lambda^2}{(p^1_- p^2_- p^3_- p^4_-)^{1/2}} \sum_{\vec{n}} \frac{F_{\{\vec{n},\vec{n}_3;\vec{n}_1\}}(p_-, p^3_-; p^1_-) F_{\{\vec{n},\vec{n}_2;\vec{n}_4\}}(p_-, p^2_-; p^4_-)}{p_- p_+ - p_- \mathcal{E}(\vec{n}, p_-)} \ , \tag{2.11}$$

with

$$\begin{aligned} p^1_- - p^3_- &= p^4_- - p^2_- = p_- \ , \\ p^1_+ - p^3_+ &= p^4_+ - p^2_+ = p_+ \ . \end{aligned} \tag{2.12}$$

In order to have a well-defined S-matrix it is necessary (though not sufficient) the above amplitudes for generic values of the parameters of external on-shell particles to be finite. For this the following two conditions should be met:
i) the propagator for the exchanged particle should never blow up, i.e. the exchanged particle should always be off-shell,[2] and
ii) the sum over all the possible excitations of the exchanged particle should be convergent.

In what follows we shall check explicitly that the above two conditions are indeed satisfied.

2.2.1 Exchanged particle is never on-shell

As we know in the usual scalar field theories as a direct result of Lorentz invariance and momentum conservation at each vertex, all the particles appearing at a vertex cannot be

[2]Of course one should note that generally when we have unstable particles it is quite possible that in a 3-particle vertex we have all three particles on-shell simultaneously. The famous example of this is the decay of Z or Higgs boson into two fermions. However, because of quantum corrections to the propagator of the unstable particle (which physically through non-relativistic Wigner-Breit formula correspond to the width of the particle) in fact the propagator does not blow up when the particle is on-shell. (For a more detailed arguments on this issue see e.g. [10].) In our cases, however, the field theories are coming as effective theories of string theory and hence in our field theory analysis the quantum corrections are not an issue. Moreover, we expect our particles at least for $M = 0$ case (corresponding to SUGRA modes) to be stable.

on-shell simultaneously. However in our problem, due to the lack of Lorentz invariance and the specific spectrum of our problem, it is not clear that this is impossible. In fact, if we only focus on the momentum conservation and put two of the particles on-shell there is nothing to prevent us from having the third particle on-shell, as well. Explicitly, let us consider the "massless" case, i.e. $M^2 = 0$ and in-going on-shell particles to have $(p^1_\pm, \vec{n}_1), (p^2_\pm, \vec{n}_2)$:

$$\begin{aligned} p^1_+ &= \mu(\sum n^1_i + \frac{D-2}{2}) , \\ p^2_+ &= \mu(\sum n^2_i + \frac{D-2}{2}) . \end{aligned} \tag{2.13}$$

Then, in the s-channel, the exchanged particle would have

$$p_+ = p^1_+ + p^2_+ = \mu \left(\sum (n^1_i + n^2_i) + D - 2 \right) . \tag{2.14}$$

Hence, for any *even* D, p_+ can be written as $\sum l_i + \frac{D-2}{2}$. One may easily check that this argument appears also working for t (or u) channels.

However, to know whether the on-shell propagation for the internal lines is actually occurring, one should check whether they really appear in the amplitudes. In other words, one should check if the corresponding $F_{\vec{n}_1,\vec{n}_2;\vec{n}}$ functions vanish or not. For that we need to work out explicit form of the integrals of Eq.(2.9). This has been done in the Appendix A, and the results for the s-channel is that

$$F_{\vec{n}_1,\vec{n}_2;\vec{n}} = 0 \quad \text{for} \quad n^1_i + n^2_i > n_i \, . \tag{2.15}$$

For the internal line in this case, one has

$$p_+ - \mathcal{E}(n_i, p_-) = \mu \sum_i (n^1_i + n^2_i - n_i) + \mu \frac{D-2}{2} + \frac{M^2}{2} \left(\frac{1}{p^1_-} + \frac{1}{p^2_-} - \frac{1}{p_-} \right) \tag{2.16}$$

When F is non-vanishing, i.e. $n^1_i + n^2_i \le n_i$, the denominator, $p_+ - \mathcal{E}(\vec{n}, p_-)$, is clearly positive and non-zero. Similarly for the t-channel,

$$p_+ - \mathcal{E}(\vec{n}, p_-) = \mu \sum_i (n^1_i - n^3_i - n_i) - \mu \frac{D-2}{2} + \frac{M^2}{2} \left(\frac{1}{p^1_-} - \frac{1}{p^3_-} - \frac{1}{p_-} \right) \tag{2.17}$$

with $p^1_- = p^3_- + p_-$. Again from the Appendix A, one has

$$F_{\vec{n},\vec{n}_3;\vec{n}_1} = 0 \quad \text{for} \quad n^1_i > n^3_i + n_i \, , \tag{2.18}$$

and for non-vanishing F $p_+ - \mathcal{E}(n_i, p_-)$ is negative definite. Therefore, the on-shell condition and the non-zero F condition cannot be satisfied simultaneously.

2.2.2 The sum over internal excitations is convergent

In order to check that the sum over the possible internal excitations is convergent, and hence the scattering amplitude is finite, we need to have F's. However, in general F's are momentum dependent and hence they are different for s, t or u channels of the four point amplitudes. Let us first consider the s-channel contributions to the scattering amplitude. Using Eq.(A.5) we find

$$F_{\vec{n}_1,\vec{n}_2;\vec{n}}(p^1_-,p^2_-;p_-) = \left(\frac{\mu p^1_- \mu p^2_-}{\pi\mu p_-}\right)^{\frac{D-2}{4}} \prod_{i=1}^{D-2}\left[\frac{1}{\sqrt{2^{n^1_i+n^2_i+n_i} n^1_i! n^2_i! n_i!}} \right.$$
$$\left. \times \sum_{l^1_i=0}^{n^1_i}\sum_{l^2_i=0}^{n^2_i} A_{l^1_i n^1_i}(\alpha) A_{l^2_i n^2_i}(\beta) \frac{2^{s_i} l^1_i! l^2_i! n_i!}{(s_i-n^1_i)!(s_i-l^1_i)!(s_i-l^2_i)!}\right] \tag{2.19}$$

where $2s_i = l^1_i + l^2_i + n_i$, $s_i \geq max\{n_i, l^1_i, l^2_i\}$, $\alpha^2 = \frac{p^1_-}{p_-}$, $\beta^2 = \frac{p^2_-}{p_-}$, and $A_{l_i n_i}(\alpha)$ functions are given in Eq.(A.4). Because of momentum conservation $\alpha^2+\beta^2 = 1$. The other vertex factor $F_{\vec{n}_3,\vec{n}_4;\vec{n}}(p^3_-,p^4_-;p_-)$ has the same expansions once p^1_-, p^2_- and $\vec{n}_1,\vec{n}_2$ are respectively replaced with p^3_-, p^4_- and $\vec{n}_3,\vec{n}_4$. We observe the following two properties of above F functions. First they are finite for any finite external momenta because $p_- = p^1_- + p^2_- = p^3_- + p^4_-$. Second they vanish whenever $n_i > n^1_i + n^2_i$ or $n_i > n^3_i + n^4_i$. Hence the range of sum on n_i in Eq.(2.9) is always bounded from above in the amplitudes with each term finite. This implies that the s-channel amplitudes are always finite once external momenta are non-vanishing.

For the t-channel we would have similar expressions for F; explicitly they are given by

$$F_{\vec{n},\vec{n}_3;\vec{n}_1}(p_-,p^3_-;p^1_-) = \left(\frac{\mu p_- \mu p^3_-}{\pi\mu p^1_-}\right)^{\frac{D-2}{4}} \prod_{i=1}^{D-2}\left[\frac{1}{\sqrt{2^{n^1_i+n^3_i+n_i} n^1_i! n^3_i! n_i!}} \right.$$
$$\left. \times \sum_{l^3_i=0}^{n^3_i}\sum_{l_i=0}^{n_i} A_{l^3_i n^3_i}(\alpha) A_{l_i n_i}(\beta) \frac{2^{s_i} l^3_i! l_i! n^1_i!}{(s_i-n^1_i)!(s_i-l^3_i)!(s_i-l_i)!}\right] \tag{2.20}$$

where $2s_i = n^1_i + l_i + l^3_i$, $s_i \geq max\{n^1_i, l_i, l^3_i\}$, $\alpha^2 = \frac{p^3_-}{p^1_-}$, $\beta^2 = \frac{p_-}{p^1_-}$ and again $\alpha^2+\beta^2 = 1$ due to the momentum conservation. The other vertex factor $F_{\vec{n},\vec{n}_2;\vec{n}_4}(p_-,p^2_-;p^4_-)$ has the same expansions as the above once p^1_-, p^3_- and $\vec{n}_1,\vec{n}_3$ are replaced with p^4_-, p^2_- and $\vec{n}_4,\vec{n}_2$, respectively.

The terms in F do not in general vanish for arbitrarily large n_i. However one can show that the sum over n_i converges, leading to finite results. Besides convergence of the sum, one should also check the behaviors of the amplitude when p_- approaches to zero. In this region, one can prove that the most singular contribution comes as

$$\mathcal{A}^t_{2\to 2} \sim \begin{cases} \frac{\lambda^2}{\sqrt{p_-}} \times \text{finite part} & \text{for D} = 3 \\ \lambda^2 \times \text{finite part} & \text{for D} > 3 \end{cases} \tag{2.21}$$

The corresponding interactions between two particles in the asymptotic region is still weak enough to allow us to define free asymptotic states as we shall see below.

To show such properties of F, let us consider the case where $\vec{n}_1$ and $\vec{n}_3$ are even with $\vec{n}_1 = 2\vec{m}_1$ and $\vec{n}_1 = 2\vec{m}_1$. Then $\vec{n}$ has to be even to have a non-zero F, which we denote by $\vec{n} = 2\vec{m}$. Then, one has

$$F_{2\vec{m},2\vec{m}_3;2\vec{m}_1}(p_-, p^3_-; p^1_-) = \left(\frac{\mu p_- \mu p^3_-}{\pi \mu p^1_-}\right)^{\frac{D-2}{4}} \prod_{i=1}^{D-2} \Bigg[\sqrt{2m_i!}\sqrt{2m^1_i!}\sqrt{2m^3_i!}2^{-m_i} \\ \times \sum_{l^3_i=0}^{m^3_i} \sum_{l_i=0}^{m_i} \frac{(-1)^{m_i-l_i+m^3_i-l^3_i}(\beta^2)^{l_i+m^3_i-l^3_i}(1-\beta^2)^{l^3_i+m_i-l_i}2^{l_i+l^3_i-m^3_i}}{(m_i-l_i)!(m^3_i-l^3_i)!(m^1_i+m^3_i-l_i)!(l^3_i+l_i-m^1_i)!(m^1_i+l_i-l^3_i)!}\Bigg] \tag{2.22}$$

with extra condition on the sum over l_i by $|m^1_i - l^3_i| \le l_i \le m^1_i + l^3_i$. From the expression one sees that each individual term is finite and behaves as $(p_-)^{\frac{D-2}{4}+\sum_i(l_i+m^3_i-l^3_i)}$, which goes to zero as one sends p_- to zero. In particular the lowest power term has a form $(p_-)^{\frac{D-2}{4}+\sum_i|m^1_i-m^3_i|}$ (since $m_i \ge |m^1_i - m^3_i|$).

Considering generic terms appearing in the $\mathcal{A}^t$ amplitudes, one finds that the term having lowest power of p_- behaves as $(p_-)^{\frac{D-4}{2}+\sum_i|m^1_i-m^3_i|+|m^4_i-m^2_i|}$ where we set $M = 0$. (When $M \neq 0$, one gets an extra power of p_- from the propagator.) Thus as claimed, one could have a singular contribution $1/\sqrt{p_-}$ for $D = 3$.

To check finiteness of the sum over m_i, let us look at large m_i behaviors. When $\beta^2 > \epsilon$ with finite ϵ, the series is convergent exponentially due to the $(1-\beta^2)^{m_i}$ term and the corresponding sum is finite in this case. Thus only potential danger may appear when $\beta^2 = p_-/p^1_-$ becomes very small. Since, the lowest power terms in p_- occur when $l^3_i = m^3_i$ we only focus on these terms in the sum. Then the expression in Eq.(2.22) becomes

$$F_{2\vec{m},2\vec{m}_3;2\vec{m}_1}(p_-, p^3_-; p^1_-) \sim \left(\frac{\mu p_- \mu p^3_-}{\pi^2 e^4 \mu p^1_-}\right)^{\frac{D-2}{4}} \prod_{i=1}^{D-2} \Bigg[(-1)^{m_i}(m_i)^{-1/4}(1-\beta^2)^{m_i} \\ \times \sum_{l_i=|m^1_i-m^3_i|}^{m^1_i+m^3_i} (-1)^{l_i}(2m_i\beta^2)^{l_i} C_{l_i m^1_i m^3_i}\Bigg] \tag{2.23}$$

where

$$C_{l_i m^1_i m^3_i} = \frac{\sqrt{(2m^1_i)!(2m^3_i)!}}{(l_i+m^1_i-m^3_i)!(l_i+m^3_i-m^1_i)!(m^1_i+m^3_i-l_i)!}\,. \tag{2.24}$$

For small p_-, one may approximate $\mu/p_- = j_0$ with some large integer j_0 and express $m_i = k_i j_0 + q_i$ with $q_i = 0, 1, \cdots, j_0 - 1$. By summing over q_i, one obtains in leading approximation

$$\mathcal{A}^t_{2\to 2} \sim \prod_{i=1}^{D-2} \left[\sum_{k_i}^{\infty} k_i^{-3/2} e^{-(\frac{\mu}{p_i^1}+\frac{\mu}{p_i^4})k_i} G_{m_i^1 m_i^3}(k_i\mu/p_i^1) G_{m_i^4 m_i^2}(k_i\mu/p_i^4) \right. \tag{2.25}$$

where the polynomial G is defined by

$$G_{m_i^1 m_i^3}(x) = \sum_{l_i=|m_i^1-m_i^3|}^{m_i^1+m_i^3} (-2x)^{l_i} C_{l_i m_i^1 m_i^3} \,. \tag{2.26}$$

The sum converges and the corresponding contributions are certainly finite even in the $p_- \to 0$ limit. For the u-channel, the effect is simply exchanging the third particle with the fourth particle. Hence their results are essentially the same as that of t-channel.

The four point scattering amplitudes essentially correspond to two body potential between particles as shown in Appendix B. From this, one may determine the region of momentum space of $(p^1_-, p^2_-, p^3_-, p^4_-)$ corresponding to the asymptotic region where particles are separated by large distance. As shown in the Appendix B, the asymptotic region corresponds to the limit where $p^3_- - p^1_-$ or $p^4_- - p^1_-$ become very small while other two independent combinations of momenta kept fixed. By the Fourier transformation of the results (2.21), we conclude that the potential $V(x_1 - x_2)$ falls off faster than or equal to $1/|x_1 - x_2|$ for $D > 3$ and $1/\sqrt{|x_1 - x_2|}$ for $D = 3$.

Here we like to comment on the strength of scalar interactions as a function of occupation numbers of the transverse harmonic oscillators. In short, for large occupation numbers, the interaction is still finite and well behaved. To see this, let us consider the characteristic vertices in Fig. 1, whose strength is basically given by $\lambda F_{\vec{n}_1,\vec{n}_2;\vec{n}}(p^1_-, p^2_-; p_-)$. To see the characteristic strength in the large occupation numbers, let us consider for example the case where $\vec{n} = \vec{n}_1 + \vec{n}_2$ and $p_- = p^1_- + p^2_-$. The explicit evaluation leads to

$$F_{\vec{n}_1,\vec{n}_2;\vec{n}}(p^1_-, p^2_-; p_-) = \prod_i \left(\frac{\mu p^1_- \mu p^2_-}{\pi \mu p_-} \right)^{1/4} \left(\frac{(n_i^1 + n_i^2)!}{n_i^1! n_i^2!} \right)^{1/2} \left(\frac{p^1_-}{p_-} \right)^{n_i^1/2} \left(\frac{p^2_-}{p_-} \right)^{n_i^2/2} \tag{2.27}$$

For fixed $N_i = n_i^1 + n_i^2$, the maximum occurs at $p^1_- = p^2_-$ and $n_i^1 = n_i^2$, in which F behaves as

$$F \sim \prod_i \left(\frac{\mu p_-}{2\pi^3} \right)^{1/4} \frac{1}{e\, n_i^{1/4}} \tag{2.28}$$

Similar trends follow for the generic F's including those involved in other channels; in the limit of the large occupation numbers they are not diverging. Hence we conclude that the strength of interaction with large external occupation number is also fine.

One may suspect that our results about the existence of S-matrix is strongly depending on the form of the ϕ^3 interactions we have turned on. Had we taken the local quartic interaction, the contribution to the four point amplitudes would behave as

$$\mathcal{A} \sim \frac{\lambda'}{\sqrt{p_-^1 p_-^2 p_-^3 p_-^4}} \int d\vec{z}\; K_{\vec{n}_1} K_{\vec{n}_2} K_{\vec{n}_3} K_{\vec{n}_4} \tag{2.29}$$

which is well behaved and finite in any case. The interactions of higher than fourth order do not contribute to the two particle amplitudes in the tree level and lead to similar result for the n_i *particle* $\rightarrow$ n_f *particle* amplitudes and hence they do not affect the existence of interaction free states in the asymptotic regions. Therefore, we may conclude that for any scalar field theory in the plane wave background with polynomial potential we have (classically) well-defined S-matrix.

3 Gauge Theory on Plane Waves

Unlike the scalar field theories, the form of the couplings are completely fixed in the gauge theories. With these strongly constrained interactions, it is interesting to ask whether the theory allows the asymptotically free states again in the 1+1 dimensional sense. The gauge theory in the plane wave background is an example of gauge theories of only massive degrees. The mechanism is completely different from the case of spontaneous symmetry breaking; rather the mass follows from the geometry. Namely the mass comes from the confining potentials in the transverse directions and the theory become effectively 1+1 dimensional as in the case of scalar theory.

It is well known that the Coulomb interactions in 1+1 dimensions do not fall off at large distance, so the asymptotically free states cannot exist. This is essentially true for any two dimensional massless field theory, such as $2D$ QED, or $2D$ gravity. Since degrees along the transverse $D-2$ directions are again confining, it is a priori not clear whether the effectively 1+1 dimensional gauge theory are free of such confining Coulomb interaction or not. Below we shall investigate such an issue via the light-cone gauge fixing. Indeed the Coulomb interaction is apparently there producing singular interactions of $(1/p_-)^2$ where p_-

is the exchanged light-cone momentum. However in the tree amplitudes one may show that all such singular contributions precisely cancel out.

There is another potential danger; on-shell photon exchanges may appear causing, further problems. Below we show in detail how one gets precise cancellation between terms appearing in the action such that the exchanged photon is always off-shell.

To avoid the unnecessary complications here we only consider the $U(1)$ gauge theory coupled to a scalar field: [3]

$$\mathcal{L} = -\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} - \frac{1}{2}g^{\mu\nu}(D_\mu\phi)^\dagger D_\nu\phi - \frac{1}{2}M^2\phi^\dagger\phi\ , \tag{3.1}$$

where $g^{\mu\nu}$ is the inverse of the metric (1.1), i.e.

$$g^{+-} = g^{-+} = -1\ , \quad g^{++} = 0\ , \quad g^{--} = \mu^2 \sum_{i=1}^{D-2} z_i^2\ , \quad g^{ij} = \delta^{ij}\ ,$$

$D_\mu\phi = \partial_\mu\phi - ieA_\mu\phi$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

3.1 Fixing the light-cone gauge

In order to study any physical processes such as scattering, one needs to fix a gauge. Because of the form of our background it turns out to be more appropriate to fix the light-cone gauge

$$A_- = 0\ . \tag{3.2}$$

As it is usual in the light-cone gauge theory analysis, since the other light-cone component A_+ is not a dynamical one, one can solve the equation of motion for A_+ and plug the solution back into the action. In this way we find an action which only involves real physical degrees of freedom. Moreover, in any axial gauge such as light-cone, ghosts are decoupled [11] and we do not need to worry about them. Imposing the $A_- = 0$ condition the Lagrangian (3.1) simplifies as

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_- A_+)^2 - \partial_- A_i \partial_i A_+ + eJ^+ A_+ \\ &\quad + \partial_+ A_i \partial_- A_i - \frac{1}{2}\mu^2 \sum_{i=1}^{D-2} z_i^2 (\partial_- A_j)^2 - \frac{1}{4}F_{ij}^2\end{aligned}$$

[3] In principle one can consider fermions. However, in order to formulate fermions in the plane wave background, one should consider the form fluxes present in the corresponding supergravity solutions. (Such fluxes lead to the fermionic mass terms.) Here we will not consider fermions. However, we believe that our result about the existence of S-matrix can be extended to those cases as well.

$$+\frac{1}{2}[(\partial_+\phi)^\dagger\partial_-\phi+(\partial_-\phi)^\dagger\partial_+\phi]-\frac{1}{2}\mu^2\sum_{i=1}^{D-2}z_i^2(\partial_-\phi)^\dagger\partial_-\phi-\frac{1}{2}(\partial_i\phi)^\dagger\partial_i\phi-\frac{1}{2}M^2\phi^\dagger\phi$$
$$-eJ_iA_i-\frac{1}{2}e^2\phi^\dagger A_iA_i\phi, \tag{3.3}$$

where

$$J^+=\frac{i}{2}\left(\phi^\dagger\partial_-\phi-(\partial_-\phi)^\dagger\phi\right)\ ,\quad J_i=\ \frac{i}{2}\left(\phi^\dagger\partial_i\,\phi-(\partial_i\,\phi)^\dagger\phi\right) \tag{3.4}$$

with $\phi=\phi_1+i\phi_2$. As we see the only A_+ dependent part is in the first line of the Lagrangian (3.3). Then the equation of motion for A_+ reads as

$$\partial_-^2A_+=\partial_-\partial_iA_i+eJ^+\ .$$

Since we consider only the field configurations with $p_-\neq 0$, ∂_-^2 is formally invertible and hence

$$A_+=\frac{1}{\partial_-}\partial_iA_i+e\frac{1}{\partial_-^2}J^+\ . \tag{3.5}$$

Inserting the above into the Eq.(3.3) we obtain the fully gauge-fixed Lagrangian

$$\begin{aligned}\mathcal{L}\ =\ &\partial_+A_i\partial_-A_i-\frac{1}{2}\mu^2(\sum_{i=1}^{D-2}z_i^2)(\partial_-A_j)^2-\frac{1}{2}(\partial_iA_j)^2\\&+\frac{1}{2}[(\partial_+\phi)^\dagger\partial_-\phi+(\partial_-\phi)^\dagger\partial_+\phi]-\frac{1}{2}\mu^2\sum_{i=1}^{D-2}z_i^2(\partial_-\phi)^\dagger\partial_-\phi-\frac{1}{2}(\partial_i\phi)^\dagger\partial_i\phi-\frac{1}{2}M^2\phi^\dagger\phi\\&-\frac{1}{2}e^2\frac{1}{\partial_-}J^+\ \frac{1}{\partial_-}J^+-e\partial_iA_i\frac{1}{\partial_-}J^+-eJ_iA_i-\frac{1}{2}e^2\phi^\dagger A_iA_i\phi,\end{aligned} \tag{3.6}$$

To run the perturbation theory machinery we need to solve the equation of motion of the quadratic parts of the action. The equations of motion for ϕ, A_i fields are exactly the same as Eq.(2.2) (of course for gauge fields we should set $M=0$), and hence the solutions for them are similar to Eq.(2.3). For the gauge field, it is

$$A^j=\sum_{\vec{n}}\frac{1}{2\pi}\int dq_+\int_0^\infty\frac{dq_-}{\sqrt{2q_-}}A^j_{\vec{n}}(q_-,q_+)\prod_{i=1}^{D-2}K_{n_i}(\sqrt{\mu q_-}z_i)e^{-i(q_-x^-+q_+x^+)}+\text{h. c.} \tag{3.7}$$

and the scalar fields may be expressed as

$$\phi_a=\sum_{\vec{n}}\frac{1}{2\pi}\int dq_+\int_0^\infty\frac{dq_-}{\sqrt{2q_-}}\phi_{a\vec{n}}(q_-,q_+)\prod_{i=1}^{D-2}K_{n_i}(\sqrt{\mu q_-}z_i)e^{-i(q_-x^-+q_+x^+)}+\text{h. c.} \tag{3.8}$$

with $a=1,2$ representing the real and imaginary parts of ϕ. Then the quadratic part of the action written in terms of Fourier modes is obtained to be

$$\mathcal{L}_0\ =\ \sum_{\vec{n}}\phi^\dagger_{(+)\vec{n}}(p_-)\left(p_+-\mathcal{E}(\vec{n},p_-)\right)\phi_{(+)\vec{n}}(p_-)+\sum_{\vec{n}}\phi^\dagger_{(-)\vec{n}}(p_-)\left(p_+-\mathcal{E}(\vec{n},p_-)\right)\phi_{(-)\vec{n}}(p_-)$$

$$+ \quad \sum_{\vec{n}} A^{j*}_{\vec{n}}(p_-)\,(p_+ - \mu\mathcal{N})\,A^{j}_{\vec{n}}(p_-) \tag{3.9}$$

where we have suppressed p_+ dependence in the component fields and introduced

$$\phi_{(\pm)\vec{n}} = \phi_{1\,\vec{n}} \pm i\phi_{2\,\vec{n}}\,. \tag{3.10}$$

The $(\pm)$ components carry respectively $(\pm e)$ charges.

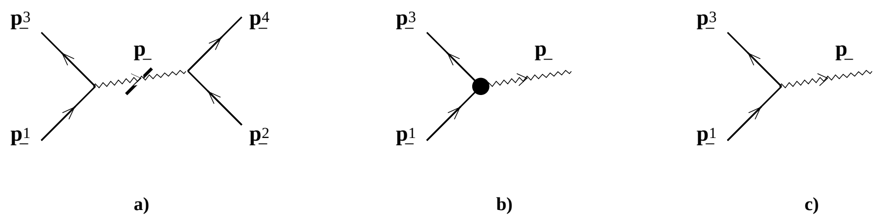

Figure 3: Vertices for the (+) charges.
a) represents "the Coulomb interaction", $\frac{ie^2(p^1_-+p^3_-)(p^4_-+p^2_-)}{(p_-)^2(2p^1_-2p^2_-2p^3_-2p^4_-)^{1/2}}\int d\vec{x}\; K_{\vec{n}_1}K_{\vec{n}_2}K_{\vec{n}_3}K_{\vec{n}_4}$ with $p_- = |p^1_- - p^3_-|$. b) corresponds to $\frac{e(p^1_-+p^3_-)}{p_-(2p^1_-2p^3_-2p_-)^{1/2}}\int d\vec{x}\; K_{\vec{n}_1}K_{\vec{n}_3}\partial_i K_{\vec{n}}$. The third diagrams is given by $-\frac{e}{(2p^1_-2p^3_-2p_-)^{1/2}}\int d\vec{x}\; K_{\vec{n}}(K_{\vec{n}_1}\partial_i K_{\vec{n}_3} - K_{\vec{n}_3}\partial_i K_{\vec{n}_1})$. These diagrams, a), b) and c) represent the first, second and third term of the third line of Eq.(3.6), respectively. There are many more diagrams differing from above only by orientation and directions of arrows. Here we do not draw all of them. In case of $(-)$ charges, $(+e)$ should be replaced by $(-e)$ accordingly.

For our later analysis it is also useful to write J^+ in terms of these Fourier modes

$$\begin{aligned} J^+ \;=\; & \sum_{\vec{n},\vec{m}} \frac{1}{(2\pi)^2}\int dp_+ dp'_+ \int_0^\infty \frac{dp_- dp'_-}{\sqrt{2p_-}\sqrt{2p'_-}}\left(\frac{i}{2}(p_- + p'_-)\right)\phi^\dagger_{(+)\vec{n}}(p_-,p_+)\phi_{(+)\vec{m}}(p'_-,p'_+) \\ & \prod_{i=1}^{D-2} K_{n_i}(\sqrt{\mu p_-}z_i)K_{m_i}(\sqrt{\mu p'_-}z_i)e^{-i(p_- - p'_-)x^- - i(p_+ - p'_+)x^+} - [(+)\to(-)] \end{aligned} \tag{3.11}$$

3.2 Evaluating $2\phi \to 2\phi$ amplitude

As in the previous section again we focus our analysis on the $2\phi \to 2\phi$ tree level scattering processes. As depicted in Fig. 4, there are five diagrams contributing to this amplitude where each diagram can be in s, t or u channel.

As we have discussed the question of having a (classically) well-defined S-matrix is now equivalent to having finite amplitudes for any generic in and out -going state. Similar to the

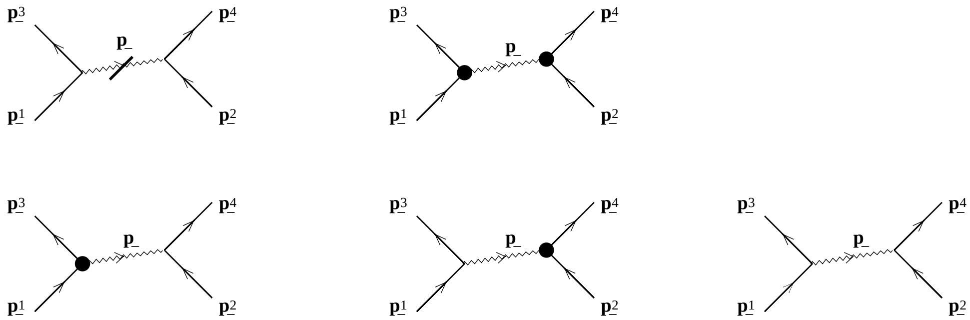

Figure 4: Four point amplitudes of $2\phi \to 2\phi$ scattering. Here we have depicted only t channel diagrams. In the $2\phi \to 2\phi$ scattering, there are also u and s channel contributions which have not been drawn here.

ϕ^3 case there are two different potential dangers. It turns out that the arguments for the convergence of the sum over all possible exchanged states is quite similar to the ϕ^3 case, hence we do not repeat them here. However as we will see, because of having spacial derivatives in the interaction vertices, there is more room for having on-shell photon exchange, which will be considered separately in the next subsection.

It is known that for the usual QED in the light-cone gauge we have $\left(\frac{1}{p_-}\right)^2$ and $\frac{1}{p_-}$ singularities in some of the graphs contributing to the $e-e$ scattering. However, summing up all the graphs, as we physically expect, these singularities cancel out and we remain with a smooth $p_- \to 0$ limit at least at the tree level. As mentioned earlier, these singular contributions come from the Coulomb type interaction of 1+1 dimension. Since the physical flux between charges spreads over all over $D-1$ dimensions in the case of flat space, these singular contributions are simply a light-cone gauge artifact. Due to the shape of plane wave geometry and the confining potential however, such a simple argument of cancellation of the flat space does not apply for gauge theory of the plane waves. Therefore, besides the two issues that are relevant for the plane wave background, we first need to clarify the $p_- \to 0$ behavior. In the following we show that the same cancellation of p_- poles is also present in the plane wave case. Note that in the plane wave case "effective mass" is also proportional to p_- and hence the $p_- \to 0$ limit should be handled with a special care.

To see the cancellation of the singular contributions explicitly, we focus on the four point interactions of (+) charges depicted in Figure 4. The first term represents instantaneous Coulomb interaction in the light cone direction. It is of order $\left(\frac{1}{p_-}\right)^2$ and its explicit expression reads as

$$\mathcal{A}_a = \frac{ie^2 W}{(p_-)^2} \int d\vec{x}\ K_{\vec{n}_1} K_{\vec{n}_2} K_{\vec{n}_3} K_{\vec{n}_4}\ , \tag{3.12}$$

where we have introduced the kinematical factor W:

$$W \equiv \frac{(p_-^1 + p_-^3)(p_-^4 + p_-^2)}{\sqrt{2p_-^1}\sqrt{2p_-^2}\sqrt{2p_-^3}\sqrt{2p_-^4}}\ . \tag{3.13}$$

The second diagram is also of order $\left(\frac{1}{p_-}\right)^2$ and expressed as

$$\mathcal{A}_b = \frac{ie^2 W}{(p_-)^2} \sum_{\vec{n}} \int d\vec{x}\ \partial_i \left(K_{\vec{n}_1} K_{\vec{n}_3}\right) K_{\vec{n}}\ \frac{1}{2p_-(p_+ - \mathcal{E}(\vec{n}, p_-))} \int d\vec{y}\ K_{\vec{n}} \partial_i \left(K_{\vec{n}_4} K_{\vec{n}_2}\right)\ . \tag{3.14}$$

Now note the identity

$$\begin{aligned} &\sum_{\vec{n}} K_{\vec{n}}(\sqrt{\mu p_-}\vec{x})\ \frac{1}{2p_-(p_+ - \mathcal{E}(\vec{n}, p_-))} K_{\vec{n}}(\sqrt{\mu p_-}\vec{y}) = \langle \vec{x}|\ \frac{1}{2p_- p_+ + \nabla^2 - (\mu p_- \vec{y})^2} |\vec{y}\rangle \\ &= \langle \vec{x}|\ \frac{1}{\nabla^2} |\vec{y}\rangle - 2p_- p_+ \langle \vec{x}|\ \frac{1}{(\nabla^2)^2} |\vec{y}\rangle + O[(p_-)^2]\,, \end{aligned} \tag{3.15}$$

where we have made expansion with respect to p_- in the second line. Using this identity, after some algebraic manipulations one can show that

$$\begin{aligned} \mathcal{A}_b = \frac{-ie^2 W}{(p_-)^2} \Bigg(&\int d\vec{x}\, K_{\vec{n}_1} K_{\vec{n}_3} K_{\vec{n}_4} K_{\vec{n}_2} \\ &+ p_- p_+ \int d\vec{x} d\vec{y}\, K_{\vec{n}_1} K_{\vec{n}_3}\ \langle \vec{x}|\ \frac{1}{\nabla^2} |\vec{y}\rangle\ K_{\vec{n}_4} K_{\vec{n}_2} + O[(p_-)^2] \Bigg)\,. \end{aligned} \tag{3.16}$$

The first term in the parenthesis is precisely canceling the contribution of $\mathcal{A}_a$. In a similar manner, one may show that the second term is canceling with the leading contributions of the third and the fourth diagrams of Figure 4, which are of order $1/p_-$. Therefore we are left only with terms that are non-negative powers of p_-.

The cancellation actually occurs in all possible tree diagram of the gauge theory. In this cancellation, the fact that the effective confining potential in the transverse directions is proportional to $(p_-)^2$ plays an important role. Because of this, the effective potential only contributes to the order of $(p_-)^2$ in the formula (3.15) and does not affect at all the cancellation of the singular interactions. If the transverse potential were too stiff, the Coulomb interaction would become essentially one dimensional and no such cancellation would occur. (With stiff enough transverse potential, fluxes cannot leak out in the transverse direction, producing one dimensional confining potential between charges.) Fortunately the potential is proportional to $(p_-)^2$ and becomes weak enough in the limit $p_- \to 0$ which corresponds to the region of the large separation of charges.

3.2.1 On-shell photon exchange is not possible

Similar to the ϕ^3 case, kinematics cannot prevent us from having on-shell photon exchange. Therefore the only way to save the theory is that the interactions are zero exactly when the parameters of the external particles allow the on-shell photon exchange. Dealing with a vector particle, $\phi\phi\gamma$ interaction terms involve space derivative of the fields and in this respect this case is different from the ϕ^3 case. In particular, noting the Eq.(A.7) of the appendix A and Eq.(2.15) or (2.18) it seems quite possible to have on-shell propagation of exchanged photon in the $D = 4,\ M = 0$ case (this will become apparent momentarily). However, noting Eq.(2.15) we see that still $D > 4$ is safe from this potential danger. The above argument can be repeated for spin S particles. In general the coupling of spin S particle to scalar fields involves S number of derivatives and according to our arguments $D \leq 2S + 2$ may be problematic. For example, for scalars coupled to six dimensional gravity ($S = 2$) on-shell propagation of exchanged gravitons is kinematically allowed. In what follows we present explicit calculations showing that the interaction exactly for the on-shell exchange of photon turns off, removing the potential danger.

For this, we consider the terms in the interaction Lagrangian which may cause a problem, i.e. those which involve spatial derivatives:

$$S_{int,\partial} = -e \int dx^+ dx^- d^{D-2}z \left(\partial_i A_i \, \frac{1}{\partial_-} J^+ + J_i A_i \right) . \tag{3.17}$$

Since we want to check whether for certain modes the above term is vanishing or not, in our computations we will drop the overall factors.

$$\begin{aligned} S_{int,\partial} \sim \int d\vec{z} \ \Bigg(& \frac{p_- + p'_-}{q_-} \partial_i K_{\vec{m}}(\sqrt{\mu q_-} z_i) K_{\vec{n}}(\sqrt{\mu p_-} z_i) K_{\vec{n}'}(\sqrt{\mu p'_-} z_i) \\ & + K_{\vec{m}}(\sqrt{\mu q_-} z_i) \partial_i K_{\vec{n}}(\sqrt{\mu p_-} z_i) K_{\vec{n}'}(\sqrt{\mu p'_-} z_i) \\ & - K_{\vec{m}}(\sqrt{\mu q_-} z_i) K_{\vec{n}}(\sqrt{\mu p_-} z_i) \partial_i K_{\vec{n}'}(\sqrt{\mu p'_-} z_i) \Bigg) . \end{aligned} \tag{3.18}$$

Note that the above expression is the one appearing in the t or u -channel where q_- is the light-cone momentum of photon and, p_- and p'_- those of in and out -going scalars, therefore $p_- = p'_- + q_-$.

We are only interested in the cases where all three particles can be on-shell at the same time. This implies that $q_+ = \sum_i m_i + \frac{D-2}{2}$. Since $p_+ = p'_+ + q_+$ due to energy conservation, for the cases of our interest (which may cause a potential danger)

$$\sum_{i=1}^{D-2} n_i - (n'_i + m_i) = \frac{D-2}{2} \ . \tag{3.19}$$

On the other hand as we discussed in the previous section, the integral

$$\int H_n(z) H_{n'}(\alpha z) H_m(\beta z) e^{-z^2} dz$$

is non-zero only for $n \leq n' + m$. Therefore, having a single derivative on K_{n_i}'s the only modes which contribute to the interactions are those with $\sum_i n_i - (n'_i + m_i) \leq 1$. Eq.(3.19) and this condition can be satisfied simultaneously only for $D = 4$ and $\sum_i n_i - (n'_i + m_i) = 1$. So, we evaluate the integral (3.18) only for the cases where one of n_i's (say n_1) is $n'_1 + m_1 + 1$ and for the rest $n_i = n'_i + m_i$, i.e.

$$\begin{aligned} S^{relevant}_{int,\partial} \sim \int d\vec{z} \quad & \Big[\partial_i K_{\vec{m}}(\sqrt{\mu q_-} z_i) \left(\frac{p_- + p'_-}{q_-} + 1 \right) K_{\vec{n}}(\sqrt{\mu p_-} z_i) K_{\vec{n}'}(\sqrt{\mu p'_-} z_i) \\ & + 2 K_{\vec{m}}(\sqrt{\mu q_-} z_i) \partial_i K_{\vec{n}}(\sqrt{\mu p_-} z_i) K_{\vec{n}'}(\sqrt{\mu p'_-} z_i) \Big]_{\sum_i n_i - (n'_i + m_i) = 1} \end{aligned} \tag{3.20}$$

Using Eqs.(A.1), (A.7) and (A.6) after some algebra we find

$$S_{int,\partial} \mid_{\sum_i n_i - (n'_i + m_i) = 1} \sim \left[\sqrt{q_-} \sqrt{\frac{q_-}{p_-}} \left(1 + \frac{p_- + p'_-}{q_-} \right) N^{-2}_{n_i} - 4 \sqrt{p_-} n_i N^{-2}_{n_i - 1} \right] \ , \tag{3.21}$$

where N_n are the normalization factors defined in (A.2). Note that to arrive at Eq.(3.21) we have dropped all the normalization and overall factors. Inserting the value of N_{n_i}'s it is readily seen that $S^{relevant}_{int,\partial} = 0$, that is, the exchanged photon is never on-shell. All the above manipulation can be repeated for the s-channel exchanges, leading to the same result. Hence, the $2\phi \to 2\phi$ amplitude (at tree level) is finite, with a smooth p_- dependence. It is straightforward to check $\phi\gamma \to \phi\gamma$ scattering and observe that it is well-behaved scattering as well. Therefore the special form of the interactions which is fixed by the gauge symmetry guarantees that we have a notion of well-defined asymptotic states, and therefore S-matrix, for gauge theories on the plane wave background for any $D > 2$.

4 Discussion and Remarks

In this work the question of existence of S-matrix for string/field theories on the plane wave background has been addressed. A priori there are several pros and cons, and hence the question asks for a direct and explicit analysis. Considering scalar field theories on a D dimensional plane wave background we argued that, because of the specific form of the geometry the field theory can effectively be considered as a $1+1$ dimensional field theory for the "harmonic oscillator" modes coming from the state of the field in the $D-2$ dimensional transverse directions. As we showed this is a general feature of any field theory on this background. Through explicit calculation of a generic four point function at tree level, we showed that these four point functions are smooth enough to allow an S-matrix interpretation. Hence we concluded that we have S-matrix formulation. Although we have not checked explicitly, we believe that the above result can be extended to the case of gravity (spin two particles).

In course of the computations in the gauge theory case (section 3.2.1) we found a non-trivial and amazing cancellation in the $\phi\phi\gamma$ interaction vertices for exactly the modes that can lead to on-shell photon exchange. Presumably this cancellation is a consequence of the gauge symmetry. It would be interesting to show explicitly how gauge symmetry is responsible for this cancellation.

In this work we only focused on the tree level checks of the existence of S-matrix. However, generally non-renormalizability of the theory may spoil the notion of unitary S-matrix at quantum (loops) level. In the field theories we have studied, we had in mind that they are coming as effective dynamics of strings in the plane wave background. Therefore the renormalizability is not an essential issue and should be addressed in the string theory level. However, one can think of these field theories independently of string theories. It would then be interesting to check whether our statement about the existence of S-matrix also holds at quantum level specifically for $D < 5$.

With the knowledge of existence of the S-matrix, the next step one may pursue will be the explicit construction of vertex operators for string theories in the plane wave background. With the vertex operators, one can in principle compute all the amplitudes using the path integration over Riemann surfaces with insertion of vertex operators. As it is known, for the flat background case working in the light-cone gauge and for $p_- = 0$, the tree and one loop diagrams may be handled without much complication using the operator formulation. However, for $p_- \neq 0$ there are severe ordering problems [12]. In the plane wave background, since the nature of physics for $p_- = 0$ and $p_- \neq 0$ are completely different, we are forced to set $p_- \neq 0$ and hence these ordering problems should show up in the vertex operators in the light-cone gauge, making the formulation of vertex operators more non-trivial. This will be the primary direction for the further studies.

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A Some useful identities about Hermite functions

The harmonic oscillator wave functions are related to the Hermite polynomials $H_n(x)$ by

$$K_n(\sqrt{\mu p_-}z) = N_n e^{-\mu p_- z^2/2} H_n(\sqrt{\mu p_-}z) \ , \tag{A.1}$$

where

$$N_n^2 = \frac{1}{2^n n!}\sqrt{\frac{\mu p_-}{\pi}} \ . \tag{A.2}$$

Using the identity [13]

$$\int e^{-x^2} H_{n+2m}(\alpha x) H_n(x) dx = \sqrt{\pi}\frac{2^n(2m+n)!}{m!}(\alpha^2-1)^m \alpha^n \ ,$$

and the orthogonality of H_n's, we have

$$H_m(\alpha z) = \sum_{k=0}^{m} A_{km}(\alpha) H_k(z) \ , \tag{A.3}$$

where $A_{km}(\alpha)$ is

$$A_{km}(\alpha) = \begin{cases} \frac{m!}{k!(\frac{m-k}{2})!}(\alpha^2-1)^{\frac{m-k}{2}}\alpha^k & m-k = \text{even}, \\ 0 & m-k = \text{odd} \end{cases} \tag{A.4}$$

Given (A.3) and using (cf. 7.375) of Ref.[13] one obtains,

$$\int e^{-z^2} H_n(z) H_m(\alpha z) H_l(\beta z) dz = \sum_{p=0}^{m}\sum_{q=0}^{l} A_{pm}(\alpha) A_{ql}(\beta) \frac{2^s\sqrt{\pi} n! p! q!}{(s-n)!(s-p)!(s-q)!} \tag{A.5}$$

where $2s = n+p+q$ and the integral is non-zero only for integer s with $s \geq max\{n,p,q\}$. A useful special case of the above integral is when $n = m+l$ where we have

$$\int e^{-z^2} H_{m+l}(z) H_m(\alpha z) H_l(\beta z) dz = \alpha^m \beta^l (m+l)! 2^{m+l}\sqrt{\pi} \ . \tag{A.6}$$

The other useful identify which has been used in the calculations of section 3, is concerning the derivative of the harmonic oscillator wave functions is $H'_n(x) = 2nH_{n-1}(x)$ which implies that

$$\frac{d}{dz}K_n(z) = \sqrt{\frac{n}{2}}K_{n-1} - \sqrt{\frac{n+1}{2}}K_{n+1}\ . \tag{A.7}$$

B General nonrelativistic field theory and asymptotic regions

The 1+1 dimensional theories of our interest has the structure of nonrelativistic field theory with multi flavor. The symplectic structure is that of field theories. The four point interaction preserves particle numbers and there is a translational invariance in x^- direction. (Below for the simplicity, we shall omit the subscript $-$ in x^- or p_-.) Here we like to consider most general such nonrelativistic field theories with quartic interactions and derive two body Schrödinger equations from it in the position space. The potential is closely related to the four point scattering amplitude and, from this, one may determine which region of momentum space corresponds to the asymptotic region where particles are separated by large distance. In short, in terms of momenta in the four point amplitudes, the asymptotic region corresponds to the limit where $p^1 - p^3$ or $p^1 - p^4$ becomes very small while other two independent combinations of momenta kept fixed.

The bosonic field theories of four point interactions is described by the following Lagrangian

$$L = i\phi^\dagger_a\dot{\phi}_a - \phi^\dagger_a H_0 \phi_a - \frac{\lambda^2}{2}\int dx_i\ \phi^\dagger_a(x_1)\phi^\dagger_b(x_2)V_{abcd}(x_1,x_2,x_3,x_4)\phi_c(x_3)\phi_a(x_4)\,, \tag{B.1}$$

where we assume the free Hamiltonian H_0 is diagonalized by the momentum eigenstates. Without loss of generality, one may take the potential satisfying the exchange symmetries,

$$V_{abcd}(x_1,x_2,x_3,x_4) = V_{bacd}(x_2,x_1,x_3,x_4) = V_{abdc}(x_1,x_2,x_4,x_3)\,. \tag{B.2}$$

Due to the translational symmetry in x direction, the potential only depends upon differences of coordinates and may be written as

$$\begin{aligned} &V_{abcd}(x_1,x_2,x_3,x_4) = V_{abcd}\left(x_1 - x_2, x_3 - x_4, \frac{x_1+x_2}{2} - \frac{x_3+x_4}{2}\right) \\ &= \int dqdq'dQ\ \tilde{V}_{abcd}(q,q',Q)\, e^{iq(x_1-x_2)}e^{-iq'(x_3-x_4)}e^{i\frac{Q}{2}(x_1+x_2-x_3-x_4)}\,. \end{aligned} \tag{B.3}$$

The tree-level four point amplitude is then given by

$$\mathcal{A}_{ab\to cd} \sim \lambda^2 \tilde{V}_{abcd}(k_1 - k_2, k_3 - k_4, k_1 + k_2)\,. \tag{B.4}$$

Upon quantization, the euqal time commutation relations are given by

$$[\phi_a(x), \phi_b^\dagger(x')] = \delta_{ab}\delta(x - x')\,. \tag{B.5}$$

Using the two body wave function defined by

$$\Phi_{ab}(x_1, x_2) \equiv \langle 0|\phi_a(x_1, t)\phi_b(x_2, t)|\Psi\rangle\,, \tag{B.6}$$

and the operator Schrödinger equation $i\dot{\phi}(x,t) = [\phi(x,t), H]$, one may get the two body Schrödinger equation,

$$i\dot{\Phi}_{ab}(x_1, x_2) = H_0\Phi_{ab}(x_1, x_2) + \lambda^2 \int dx_3 dx_4 V_{abcd}(x_1, x_2, x_3, x_4)\Phi_{cd}(x_3, x_4)\,. \tag{B.7}$$

In the eigenbasis of total momentum $\Phi_{ab}(x_1 - x_2, Q)e^{i\frac{Q}{2}(x_1+x_2)}$, the Schrödinger equation become

$$i\dot{\Phi}_{ab}(x_r, Q) = H_0\Phi_{ab}(x_r, Q) + \lambda^2(2\pi)^2 \int dq dq'\; e^{iqx_r}\, \tilde{V}_{abcd}(q, q', Q)\Phi_{cd}(q', Q)\,, \tag{B.8}$$

where $x_r = x_1 - x_2$ and

$$\Phi_{ab}(q, Q) = \frac{1}{2\pi}\int dx_r \Phi_{ab}(x_r, Q)\, e^{-iqx_r}\,. \tag{B.9}$$

Let us further introduce $\bar{V}_{abcd}(y, y', Q)$ by

$$\tilde{V}_{abcd}(q, q', Q) = \frac{1}{8\pi^2}\int dy dy'\; \bar{V}_{abcd}(y, y', Q)\, e^{-\frac{iy}{2}(q-q')}e^{-i\frac{y'}{2}(q+q')}\,. \tag{B.10}$$

It is then straightforward to check that the two body Schrödinger equation becomes

$$\begin{aligned}(i\partial_t - H_0)\Phi_{ab}(x_r, Q) &= 2\pi\lambda^2 \int dy'\, \bar{V}_{abcd}(2x_r - y', y', Q)\Phi_{cd}(x_r - y', Q) \\ &= 2\pi\lambda^2 \int dy\, \bar{V}_{abcd}(y,\, 2x_r - y\,, Q)\Phi_{cd}(y - x_r\,, Q)\,,\end{aligned} \tag{B.11}$$

From this expression, it is clear that the large separation between particles ($x_r \to \infty$) corresponds to the region where y or y' of the potential becomes large. This in turn implies that $q - q' = k_1 - k_3$ or $q + q' = k_1 - k_4$ becomes small where we have used the momentum conservation. Hence the asymptotic region corresponds to the momentum space region of four point amplitudes in which $k_1 - k_3$ or $k_1 - k_4$ become small.

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